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λ_1, Isoperimetric Inequalities for Graphs, and Superconcentrators

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A general method for obtaining asymptotic isoperimetric inequalities for families of graphs is developed. Some of these inequalities have been applied to functional analysis. This method uses the second smallest eigenvalue of a certain matrix associated with the graph and it is the discrete version of a method used before for Riemannian manifolds. Also some results are obtained on spectra of graphs that show how this eigenvalue is related to the structure of the graph. Combining these results with some known results on group representations many new examples are constructed explicitly of linear sized expanders and superconcentrators. © 1985 Academic Press, Inc.

1. INTRODUCTION

Suppose $l \geqslant 1$ and let $G = G_l = (V_l, E_l) = (V, E)$ be the graph of the l-dimensional cube. The vertices of G are the 2^l binary vectors of length l and two vertices are adjacent iff they differ in exactly one coordinate. The distance between any two vertices is thus the number of coordinates in which they differ. For $v \in V_l$ and $d \geqslant 1$ let $B(v, d)$ denote a Hamming ball with center v and radius d, i.e., a set consisting of all vertices of G whose distance from v is less than d and *some* vertices of G whose distance from v is d. Let $\bar{0}$ and $\bar{1}$ denote the all-zeros and the all-ones vertices of G. A well-known theorem of Harper ([22], see also [17] for a simple proof and [35] for a generalization) asserts that if A, C are two subsets of vertices of

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G and $\rho = \rho(A, C)$ is the distance between them, then there exist two Hamming balls $B_1 = B(\bar{0}, d_1)$ and $B_2 = B(\bar{1}, d_2)$ such that $|B_1| = |A|$, $|B_2| = |C|$ and the distance between B_1 and B_2 is at least ρ. This theorem, which gives the precise solution to the isoperimetric problem for G_l is used in [4] to obtain the following asymptotic result.

PROPOSITION 1.1. *Suppose* $A, C \subseteq V$ $\mu(A) = |A|/|V| \geqslant \frac{1}{2}$ *and* $\mu(C) = |C|/|V|$. *If* $\rho = \rho(A, C)$ *then* $\mu(C) \leqslant \exp(-[2\rho^2/l])$.

*N*ote that l is the diameter of G_l and thus, roughly speaking, Proposition 1.1 asserts that a "small" neighbourhood of any set of half of the vertices of G_l contains "almost all" the vertices (e.g., if $\rho(A, C) \geqslant 10\sqrt{l}$ then $\mu(C) \leqslant \exp(-200)$). Results analogous to Proposition 1.1 for many continuous metric spaces (such as the Euclidean spheres S^n—[24] and the toruses T^n—[21]) are well known. Similar results are also known for some discrete metric spaces (such as the permutation groups π_n—see [28], and more generally [32]. Here π_n is the graph whose vertices are all permutations on $\{1, 2, ..., n\}$ and two permutations τ, σ are adjacent iff $\tau\sigma^{-1}$ is a transposition.) These results motivated the definition of the so-called *Levy family* of metric probability spaces, (See [21, 29].) Here we consider this notion for families of graphs.

DEFINITION 1.2. Let $G_i = (V_i, E_i)$ be a family of graphs and let ρ_i be the diameter of G_i. For $A \subset V_i$ and $\rho > 0$ put $A_\rho = \{v \in V_i : \rho(v, A) \leqslant \rho\}$, where $\rho(v, A)$ is the distance (in G_i) between v and A. Define $1 - \alpha(i, \rho) = \min\{\mu(A_\rho)\ (= |A_\rho|/|V_i|) : A \subset V_i,\ \mu(A) \geqslant \frac{1}{2}\}$. The family $\{G_i\}$ is a *Levi family* if for every $\varepsilon > 0$,

$$\lim_{i \to \infty} \alpha(i, \varepsilon\rho_i) = 0.$$

It is a *concentrated* (*Levy*) *family* if there exist constants c, h such that for every i,

$$\alpha(i, \rho) \leqslant c \cdot \exp(-[h\rho/\sqrt{\rho_i}]).$$

Proposition 1.1 easily implies that the family of cubes is a concentrated family. (Actually, it implies a stronger assertion—that this is a normal Levy family (see [5] for the definition and for some examples).) This fact has many applications in studying normed spaces, since it supplies a measure of the set where a function from V_i to the real numbers is ε-close to its median in terms of its modulus of continuity.

Let π_n denote the permutation graph defined above. Maurey [28] used Martingale theory to prove that the family $\{\pi_n\}$ is concentrated. He also used this result to obtain some applications in functional analysis. His method, which was extended by Schechtman [32], can sometimes solve the asymptotic isoperimetric problem (i.e., supply results similar to Proposition 1.1), even when the precise solution is not known. All these results have applications in the asymptotic theory of normed spaces [29], where of course, the asymptotic results are sufficient.

In this paper we develop a general method to prove that various families of graphs are concentrated.. Our method uses $\lambda_1 = \lambda_1(G)$, the second smallest eigenvalue of a well-known matrix associated with the graph G, and it is in fact the discrete version of the "Laplace operator" approach used in [21] for smooth Riemannian manifolds.

We also obtain some results on spectra of graphs that show how $\lambda_1(G)$ is related to the structure of G. In [13, p. 269] (see also [9]) the authors note that in a table of 3-regular graphs on $\leqslant 14$ vertices there seems to appear a strong correspondence between λ_1 and the "shape" of the graph which still lacks a precise formulation. The results proved in Section 2 (especially Theorems 2.5, 2.6, 2.7) seem to offer such a formulation.

Our study of λ_1 is related to the well-known problem of explicit construction of linear sized expanders and superconcentrators (see [27, 19]). In Section 4 we show how to construct expanders from graphs with "large" λ_1. Since λ_1 can be computed efficiently this construction supplies a lower bound on the quality of the expanders obtained. Combining this method with results of Kazhdan [23] on group representations we construct explicitly many new examples of linear sized expanders and superconcentrators. The construction is similar to that of Margulis [27] but is somewhat more general, supplies more "natural" examples (double covers of certain Cayley graphs), and reveals the connection between λ_1, the concentration property and expanders.

Our paper is organized as follows. In Section 2 we develop our main tools and obtain several results on graph spectra. In Section 3 we combine these with some known results about spectra of graphs and prove that various families of graphs are concentrated. In Section 4 we show how our results together with those of Kazhdan on group representations supply an explicit construction of linear sized expanders and superconcentrators. Finally, in Section 5 we discuss a possible application of our results to combinatorial group theory.

We would like to thank M. Gromov for fruitful discussions. The combinatorial part 9.1 of his paper [20] was the starting point of this paper. We would also like to thank D. Kazhdan for his major contribution to Section 4.

2. The Main Tools

Let $G = (V, E)$ be a connected graph on $|V| = n$ vertices, and let D be an orientation of G. Let $C = C_D = (c_{e,v})_{e \in E, v \in V}$ be the incidence matrix of D, i.e., a matrix with $|E|$ rows indexed by the edges of D and $|V|$ columns indexed by the vertices of D in which

$$\begin{aligned} c_{e,v} &= 1 && \text{if } v \text{ is the head of } e, \\ &= -1 && \text{if } v \text{ is the tail of } e, \\ &= 0 && \text{otherwise.} \end{aligned}$$

Define $Q = Q_G = C^{\mathrm{T}} \cdot C$. One can easily check that $Q = \operatorname{diag}(d(v))_{v \in V} - A_G$, where $d(v)$ is the degree of the vertex $v \in V$ and A_G is the adjacency matrix of G. Therefore Q is independent of the orientation D of G.

Let $L^2(V)$ ($L^2(E)$) denote the space of real valued functions on V (on E) with the usual scalar product (f, g) and the usual norm $\|f\| = \sqrt{(f, f)}$ induced by it.

Consider the quadratic form (Qf, f) defined on $f \in L^2(V)$. Clearly this is a positive semidefinite form. If we let e^+ and e^- denote the head and the tail of the edge e of D it is readily seen that

$$(Qf, f) = (Cf, Cf) = \sum_{e \in E} (f(e^+) - f(e^-))^2.$$

Since G is connected we conclude that $(Qf, f) \geqslant 0$ for all $f \in L^2(V)$ and equality holds iff f is a constant. Let $0 = \lambda_0 < \lambda_1 = \lambda_1(G) \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_{n-1}$ be the eigenvalues of Q, each appearing in accordance with its multiplicity. By the well-known Rayleigh's principle if $f \in L^2(V)$ is orthogonal to the constants ($=$ eigenvectors corresponding to $\lambda_0 = 0$) then

$$(Qf, f) \geqslant \lambda_1 \|f\|^2. \tag{2.1}$$

The matrix Q is commonly used in graph theory in finding the number of spanning trees of G (see, e.g., [11, Chap. 6]), and its spectrum was investigated by various authors (see [7, 15, and 18]). Fiedler ([18], see also [13, pp. 265–266]) called $\lambda_1(G)$ the algebraic connectivity of G and investigated some of its properties. It seems that graphs with large λ_1 tend to have large girth and connectivity. The main results of this section (whose exact formulation is stated in Theorems 2.5, 2.6, and 2.7) show that large λ_1 also implies a small diameter and is related to the important concentration property described in Section 1.

It is helpful to consider the operator $Q: L^2(V) \to L^2(V)$ as the (minus)

Laplace operator for the graph $G=(V, E)$, and to consider the operator $C: L^2(V) \to L^2(E)$ as the gradient operator for G. In this setting, the main difference between Theorems 2.5 and 2.6 below and their analytic analogue in [21] is the factor d corresponding to the maximum degree of a vertex of G. This difference seems to arise from the fact that there is no analogue to the unique direction of the gradient for the discrete case.

Returning to our graph $G=(V, E)$, let A and B be two disjoint subsets of V, let ρ be the distance (in G) between them and put $a=|A|/n$, $b=|B|/n$. Let E_A (E_B) be the set of edges of G with both endpoints in A (in B). The following lemma is the main tool in proving the main results of this section. As noted by one of the referees an analytic result related to it was proved by Cheeger [14].

LEMMA 2.1.

$$\lambda_1 \cdot n \leqslant \frac{1}{\rho^2}\left(\frac{1}{a}+\frac{1}{b}\right)(|E|-|E_A|-|E_B|).$$

Proof. Define a function $g \in L^2(V)$ by

$$g(v)=\frac{1}{a}-\frac{1}{\rho}\left(\frac{1}{a}+\frac{1}{b}\right)\min(\rho(v, A), \rho),$$

where $\rho(v, A)$ is the distance from v to A. If $v \in A$ then $g(v)=1/a$ and if $v \in B$ then $g(v)=-1/b$. Clearly if u, $v \in V$ are adjacent then $|g(u)-g(v)| \leqslant 1/\rho(1/a+1/b)$ and thus

$$|(Cg)(e)| \leqslant \frac{1}{\rho}\left(\frac{1}{a}+\frac{1}{b}\right) \qquad \text{for all edges } e \in E.$$

Define $\alpha=(1/n)\sum_{v \in V} g(v)$ and put $f=g-\alpha$. Obviously, $\sum_{v \in V} f(v)=0$ and thus inequality (2.1) holds for f. Therefore

$$\begin{aligned}
\lambda_1 \cdot n\left(\frac{1}{a}+\frac{1}{b}\right) &\leqslant \lambda_1\left(\left(\frac{1}{a}-\alpha\right)^2 \cdot n \cdot a+\left(\frac{1}{b}+\alpha\right)^2 n \cdot b\right) \\
&= \lambda_1 \sum_{v \in A \cup B} f^2(v) \leqslant \lambda_1 \sum_{v \in V} f^2(v) \\
&= \lambda_1 \cdot \|f\|^2 \leqslant (Qf, f)=(Cf, Cf) \\
&= \sum_{e \in E} (f(e^+)-f(e^-))^2=\sum_{e \in E} (g(e^+)-g(e^-))^2
\end{aligned}$$

$$= \sum_{e \in E - (E_A + E_B)} (g(e^+) - g(e^-))^2$$

$$\leqslant \frac{1}{\rho^2}\left(\frac{1}{a}+\frac{1}{b}\right)^2 \cdot (|E| - |E_A| - |E_B|).$$

The desired result follows. ∎

Remark 2.2. If A, B are disjoint subsets of V and $a = |A|/n = b = |B|/n = \frac{1}{2}$, then the set of all edges joining a vertex of A to a vertex of B is called a *bisector* of G. There are several papers dealing with the cardinality of the minimal bisector of G. Lemma 2.1 (with $\rho = 1$) shows that the cardinality of any bisector of G is at least $\lambda_1 \cdot n/4$. This inequality holds as an equality for the graph of the l-dimensional cube for all $l \geqslant 1$. (See Remark 3.3 in Section 3.) More general results about the connection between bisectors and the spectrum of Q appear in [16].

Remark 2.3. Lemma 2.1 shows that $\lambda_1 \leqslant (n/(n-1)) \min\{d(v) \colon v \in V\}$. Indeed, let $u \in V$ satisfy $d(u) = \min(d(v) \colon v \in V)$, and apply Lemma 2.1 with $A = \{u\}$, $B = V \backslash \{u\}$, $\rho = 1$, and $|E| - |E_A| - |E_B| = d(u)$ to get the last inequality. This inequality was proved in [18] in a different method.

Remark 2.4. Bussemaker, Cvetković, and Seidel [10] (see also [13, p. 115]) used an interlacing theorem of Haemers to prove the following result:

Let G be a d-regular graph on n vertices and let λ_1 be as above. If G_1 is an induced subgraph of G on a set V_1 of n_1 vertices with average degree d_1 then

$$d_1 \leqslant \frac{n_1 \lambda_1}{n} + d - \lambda_1.$$

Using Lemma 2.1 we obtain an alternative elementary proof of this result. Indeed, Lemma 2.1 with $A = V_1$, $B = V - V_1$, and $\rho = 1$ shows that at least $\lambda_1 n_1 (n - n_1)/n$ edges join vertices of V_1 to vertices of B. Thus, the sum of degrees (in G_1) of vertices of G_1 is at most $dn_1 - \lambda_1 n_1 (n - n_1)/n = n_1(n_1 \lambda_1/n + d - \lambda_1)$ and the average degree is at most $n_1 \lambda_1/n + d - \lambda_1$, as needed.

THEOREM 2.5. *Let G, A, B, a, b, ρ, and λ_1 be as above, and let d be the maximum degree of a vertex of G. If $\rho > 1$ then*

$$b \leqslant \frac{1-a}{1 + (\lambda_1/d)\, a \rho^2}. \tag{2.2}$$

Proof. Since $\rho > 1$ every edge in the set $E-(E_A \cup E_B)$ is incident with at least one of the $n-na-nb$ vertices of the set $V-(A \cup B)$. Thus

$$|E|-|E_A|-|E_B| \leqslant n(1-a-b)\,d.$$

This and Lemma 2.1 imply

$$\lambda_1 \cdot n \leqslant \frac{1}{\rho^2}\left(\frac{1}{a}+\frac{1}{b}\right)\cdot n(1-a-b)\,d \leqslant \frac{1}{\rho^2 ab}\cdot n(1-a-b)\,d.$$

The desired result follows. ∎

THEOREM 2.6. *Let G, A, B, a, b, λ_1, and d be as above and assume that the distance between A and B is greater than $\rho \geqslant 1$. (We do not assume here that ρ is an integer.) Then*

$$b \leqslant (1-a)\exp(-\ln(1+2a)[\sqrt{(\lambda_1/2d)}\,\rho]). \tag{2.2}$$

Proof. Define $\mu = \sqrt{2d/\lambda_1}$. By Remark 2.3, $\mu \geqslant 1$ (assuming $n \geqslant 2$). For a subset F of V and a positive real r, define $F_r = \{v = V\colon d(v, F) \leqslant r\}$ and $f_r = |F_r|/n$. Define also $k = [\rho/\mu]$. For $0 \leqslant j < k$ the distance s between the sets $A_{j\mu}$ and $V - A_{(j+1)\mu}$ is strictly greater than $\mu \geqslant 1$. Therefore, by Theorem 2.5,

$$\begin{aligned} 1-a_{(j+1)\mu} &\leqslant (1-a_{j\mu})\cdot\frac{1}{1+(\lambda_1/d)\,a_{j\mu}\cdot s^2} \leqslant (1-a_{j\mu})\frac{1}{1+(\lambda_1/d)\,a\mu^2} \\ &= (1-a_{j\mu})\cdot\frac{1}{1+2a} \end{aligned}$$

for all $0 \leqslant j < k$.

Multiplying these k inequalities we conclude that

$$1-a_{k\mu} \leqslant (1-a)\cdot\left(\frac{1}{1+2a}\right)^k = (1-a)\exp(-ln(1+2a)\cdot k).$$

Since $B \subseteq V - A_{k\mu}$ the last inequality implies (2.2). ∎

THEOREM 2.7. *Let $G=(V,E)$ be a connected graph on $|V| = n > 1$ vertices, with maximal degree d, and put $\lambda = \lambda_1(G)$. Then the diameter of G is at most*

$$2[\sqrt{(2d/\lambda)}\log_2 n].$$

Proof. Define $\rho = \sqrt{(2d)/\lambda}\log_2 n$. We first show that if A is any set of at least half the vertices of V then $A_{[\rho]} = V$. Indeed, let B denote the set of all vertices of G whose distance from A is more than ρ. Put $a=|A|/n \geqslant \frac{1}{2}$, $b=|B|/n$. By Theorem 2.6,

$$b \leqslant \tfrac{1}{2}\exp(-ln\, 2[\sqrt{\lambda/(2d)}\,\rho]) < 1/n,$$

and thus $B=\emptyset$. Thus $A_{[\rho]}=V$. Next we show that if $v \in V$ then $|\{v\}_{[\rho]}| \geqslant n/2$. Indeed, suppose this is false and define $A=V\backslash\{v\}_{[\rho]}$. Then $|A|/n > \frac{1}{2}$ and thus $A_{[\rho]}=V$. In particular, the distance between v and A is at most $[\rho]$, contradicting the definition of A. Thus $|\{v\}_{[\rho]}| \geqslant n/2$ and $(\{v\}_{[\rho]})_{[\rho]}=V$, which is the desired result. ∎

Remark 2.8. In Section 4 we construct, for a fixed $\varepsilon>0$, a sequence of 4-regular graphs $G_i=(V_i, E_i)$, with $|V_i| \to \infty$ and $\lambda_1(G_i) \geqslant \varepsilon$. By Theorem 2.7, the diameter of G_i is at most $c_1 \log_2 |V_i|$, where $c_1=c_1(\varepsilon)$ is independent of $|V_i|$. However, since G_i is 4-regular the number of vertices in V_i whose distance from a fixed $v \in V_i$ is $\leqslant \rho$ is at most $4+4\cdot 3+4\cdot 3^2+4\cdot 3^{\rho-1}$. Therefore, the diameter of G_i is at least $c_2 \log_2 |V_i|$. This shows that Theorem 2.7 is in a sense, best possible (and hence so is Theorem 2.6).

3. Concentrated Families of Graphs

In order to apply Theorem 2.6 one has to find methods to compute or estimate $\lambda_1=\lambda_1(G)$ for various graphs G. In this section we describe such methods and combine them with Theorem 2.6 to prove that some families of graphs are concentrated.

The first simple method deals with cartesian products of graphs. Recall that if $G_1=(V_1, E_1)$ and $G_2=(V_2, E_2)$ are graphs, the *cartesian product* of them is a graph $G=(V, E)$, where $V=V_1 \times V_2$ and (v_1, u_1), $(v_2, u_2) \in V$ are adjacent if $v_1=v_2$ and u_1 is adjacent (in G_2) to u_2 or if $u_1=u_2$ and v_1 is adjacent (in G_1) to v_2. Let Q_1, Q_2, and Q be the Q-matrices defined in Section 2 of G_1, G_2, and G, respectively. One can easily check that $Q=Q_1 \times I+I \times Q_2$, where $\times$ is the well-known kronecker product. This means that if $\{\lambda_i\}$ and $\{\mu_j\}$ are the sets of eigenvalues of Q_1 and Q_2, respectively, then the set of eigenvalues of Q is $\{\lambda_i+\mu_j\}$. In particular, if G_1 and G_2 are connected, then

$$\lambda_1(G_1 \times G_2)=\min(\lambda_1(G_1), \lambda_1(G_2)).$$

This proves the only nontrivial part of the following proposition.

Proposition 3.1. *For $1 \leqslant i \leqslant l$ let G_i be a connected graph with diameter ρ_i, maximal degree d_i, and put $\lambda_i=\lambda_1(G_i)$. Let G be the cartesian product of* *the l G_i-s. Then the diameter of G is $\sum_{i=1}^{l}\rho_i$, its maximal degree is $\sum_{i=1}^{l}d_i$, and $\lambda_1(G)=\min_{1\leqslant i\leqslant l}\lambda_i$.*

Combining this proposition with Theorem 2.6 we obtain

THEOREM 3.2. *Let H be any fixed connected graph. For $i\geqslant 1$ let G_i be the cartesian product of i copies of H. Then $\{G_i\}$ is a concentrated family of graphs.*

Remark 3.3. Clearly Theorem 2.6 supplies a constant h for which the family $\{G_i\}$ is concentrated. For example, we can use it to show that if $G=(V,E)$ is the l-dimensional cube (which is the cartesian product of l copies of K_2) then $\lambda_1(G)=2$ and thus if $A, C\subseteq V$ satisfy $\mu(A)\geqslant\frac{1}{2}$ and $\rho(A,C)>\rho$ then $\mu(C)\leqslant\frac{1}{2}\exp(-\ln 2\,[\rho/\sqrt{l}])$. (Proposition 1.1 contains, of course, a better result for this case.)

It is also worth noting that the results of [32] also imply that $\{G_i\}$ is a Levi family with $\alpha(i,\rho)\leqslant 2\exp(-\frac{1}{64}[\rho^2/i])$ (and is thus certainly a concentrated family).

Next we consider regular graphs. Clearly if G is regular there is a simple relationship between the spectrum of the adjacency matrix of G (which is commonly called the spectrum of G) and the spectrum of the matrix Q_G. In particular, if G is d-regular then $\lambda_1(G)$ is just the difference between d and the second largest eigenvalue of G. There are many results about the eigenvalues of graphs and here we use some of them to construct more examples of concentrated families of graphs.

THEOREM 3.4. (i) *For $k>1$ let S_k be a set of cardinality $2k-1$. The odd graph O_k is a graph whose vertices correspond to the subsets of S_k of cardinality $k-1$ and two vertices are joined iff the corresponding subsets are disjoint (see [11, p. 56]). The family $\{O_k\}_{k=1}^{\infty}$ is concentrated.*

(ii) *If $\{G_i\}_{i=1}^{\infty}$ is a concentrated family of regular graphs then so is the family $\{L(G_i)\}_{i=1}^{\infty}$ of their line graphs. In particular $\{L(O_k)\}_{k=1}^{\infty}$ is concentrated.*

Proof. (i) One can easily check that O_k is k-regular with diameter $k-1$. By [11, p. 145], $\lambda_1(O_k)=2$. This and Theorem 2.6 imply the desired result.

(ii) Is an easy consequence of the definitions. If we want to apply Theorem 2.6 directly to $L(G_i)$ to obtain some constant h for which $L(G_i)$ is concentrated we can use the result of Sachs (see, e.g., [11, p. 19]) that implies that $\lambda_1(G)=\lambda_1(L(G))$ for every regular graph G. ∎

We close this section with a brief discussion of Cayley graphs. Let H be a finite group with a generating multiset δ satisfying $\delta=\delta^{-1}$. The *Cayley* *graph* $G = G(H, \delta)$ with vertex set H is defined as follows: The vertices u and v are connected by s parallel edges, where s is the multiplicity of uv^{-1} in δ. (Usually we assume that δ is a set and $1 \notin \delta$. In this case G is a simple Cayley graph.) Clearly $G(H, \delta)$ is $|\delta|$ regular. Lovaśz ([25], see also [26 and 8] for a simplified formula) found a formula for the power sums of the eigenvalues of simple Cayley graphs in terms of characters of H. For an abelian H this formula implies

$$\lambda_1(G) = |\delta| - \max_{\chi} \left\{ \sum_{s \in \delta} \chi(s) \colon \chi \text{ is a nontrivial multiplicative character of } H \right\}. \tag{3.1}$$

This formula easily implies that for every simple Cayley graph G over the group $H = (Z_2)^l$, $\lambda_1(G)$ is an even integer. In particular we can use (3.1) to show that for the l-dimensional cube $\lambda_1 = 2$ and thus the set of l-dimensional cubes is concentrated (see also Remark 3.3; Gromov [20, p. 116] gave an equivalent definition for λ_1 and computed it for the cube using some variational inequalities).

In the next section we show how the theory of group representations can be used to prove that certain families of Cayley graphs of nonabelian groups are concentrated in a very strong sense.

4, Expanders and Superconcentrators

An (n, k)-*superconcentrator* is a directed graph with n inputs, n outputs, and at most $k \cdot n$ edges, such that for every $1 \leqslant r \leqslant n$ and every two sets of r inputs and r outputs there are r vertex disjoint paths connecting the two sets. A family of linear superconcentrators of density k is a set of $(n_i, k + o(1))$ superconcentrators, with $n_i \to \infty$, as $i \to \infty$. The problem of constructing such a family is well known (see [19] and its references.) Following [19] we let an (n, k, c)-*expander* denote a bipartite graph with n inputs, n outputs, and at most kn edges, such that for every subset A of inputs $|N(A)| \geqslant (1 + c(1 - |A|/n)) \cdot |A|$, where $N(A)$ denotes the set of all neighbors of vertices of A. In [19 and 33] it is shown how expanders can be used to construct superconcentrators. The construction of [33] implies that in order to construct a family of linear superconcentrators it is enough to construct a family of linear expanders, i.e., to construct for some fixed k and $c > 0$ a family of (n_i, k, c)-expanders (with $n_i \to \infty$ as $i \to \infty$). Such a family is also useful in a recent parallel sorting network described in [1].

Although it is rather easy to prove the existence of a family of the desired type using probabilistic methods, an explicit construction is difficult. In a most interesting paper [27] Margulis shows such a construction. He uses the theory of group representations to prove that it has the desired property. A variation of his construction is given in [6]. Gabber and Galil [19] give another variation of Margulis' construction and a charming proof that it has the desired property using Fourier analysis. All these constructions are basically the one given in [27], with small variations. However, the proof of [19], unlike that of [27], supplies a lower bound to the amount of expansion c, and this enables the authors to construct a family of linear superconcentrators of density ≈ 272.

Here we show how information about λ_1, together with some results on group representations, can be used to construct many new examples of families of linear expanders. Other relations between eigenvalues of certain matrices associated with graphs and their expansions properties appear in [34, 2] (see also [12]). Our method here is similar to that of [27], but is somewhat more general. It also reveals the connection between expanders and the concentration property. Combining our methods with those of [19 and 33] we can construct a family of linear superconcentrators of density ≈ 71, and a family of acyclic linear superconcentrators of density ≈ 158. Both results improve significantly the previous best known constructions. This will appear in [3]. (We would like to thank D. Kazhdan who showed us how his results on group representations can be used here.)

DEFINITION 4.1. An *(n, k, ε)-enlarger* is a k-regular graph G on n vertices with $\lambda_1(G) \geqslant \varepsilon$.

DEFINITION 4.2. Let $G = (V, E)$ be a graph on a set of n vertices $V = \{v_1, v_2, ..., v_n\}$. The *extended double cover of* G is a bipartite graph H on the sets of inputs $X = \{x_1, x_2, ..., x_n\}$ and outputs $Y = \{y_1, y_2, ..., y_n\}$ in which $x_i \in X$ and $y_j \in Y$ are adjacent iff $i = j$ or $v_i v_j \in E$.

Our basic observation is

THEOREM 4.3. *Let $G = (V, E)$ be an (n, k, ε)-enlarger and let H be its extended double cover. Then H is an $(n, k+1, c)$-expander for $c = 4\varepsilon/(k + 4\varepsilon)$.*

Proof. Let A be a subset of inputs of H and let $\bar{A}$ be the corresponding subset of V. Clearly $|A| = |\bar{A}|$ and the cardinality of $N(A)$ is precisely that of the set of all vertices of G whose distance from $\bar{A}$ is <2. By Theorem 2.5 the set of all other vertices of G is of size at most

$$n \cdot (1-a)\Big/\left(1 + \frac{\lambda_1(G)}{k} a \cdot 2^2\right) \leqslant n(1-a)\Big/\left(1 + 4\frac{\varepsilon}{k} a\right),$$

where $a = |\bar{A}|/n (= |A|/n)$. Therefore

$$|N(A)| \geqslant n - n(1-a)/\left(1 + 4\frac{\varepsilon}{k} a\right) = \left(1 + \frac{4\varepsilon}{k + 4\varepsilon a}(1-a)\right) na$$

$$\geqslant \left(1 + \frac{4\varepsilon}{k + 4\varepsilon}(1 - |A|/n)\right) \cdot |A|,$$

which is the desired result. ∎

Remark 4.4. Given a k-regular graph G on n vertices, there are several well-known efficient algorithms to compute $\lambda_1(G)$. (See, e.g., [31]). Therefore, we can determine, given $\varepsilon > 0$, if G is an (n, k, ε) enlarger. If so, Theorem 4.3 enables us to construct the corresponding expander. In contrast, there is no known efficient algorithm to decide if a given bipartite graph is an (n, k, c)-expander.

It is worth noting that a properly stated converse of Theorem 4.3 is also true. This follows from a discrete version of Cheeger's result [14] and will appear somewhere else.

By Theorem 4.3, in order to construct a family of linear expanders it is enough to construct a family of linear enlargers, i.e., to construct, for some fixed k and $\varepsilon > 0$, a family of (n_i, k, ε)-enlargers with $n_i \to \infty$. Note that by Theorem 2.6 such a family is a concentrated family of graphs, in a very strong sense. We proceed to show how to construct such a family using the results of Kazhdan [23] on property (T).

DEFINITION 4.5. A unitary representation π of a group H in the space $V = V_\pi$ is called *essentially nontrivial* if for any vector $0 \neq v \in V$, there exists an $h \in H$ such that $\pi(h) v \neq v$. (I.e., the unit representation is not a subrepresentation of π.)

DEFINITION 4.6 ([23], *see also* [36, p. 406]). Let H be a locally compact group. H has *property* (T) if there exist $\varepsilon > 0$ and a compact $K \subset H$ such that for every essentially nontrivial unitary representation π of H in $V = V_\pi$ and for every unit vector $y \in V$ there exists an $h \in K$ such that

$$|(\pi(h) y, y)| < 1 - \varepsilon.$$

The following lemma is a consequence of the last definition. Its proof can be easily deduced from [27 (English version), p. 330].

LEMMA 4.7. *If H is a discrete group having property (T), and S is a set of generators of H, then there exists a constant $\varepsilon > 0$ such that for every* *essentially nontrivial unitary representation* π *of* H *in* $V = V_\pi$ *and for every unit vector* $y \in V$ *there exists an* $s \in S$ *such that* $|(\pi(s)\, y, y)| < 1 - \varepsilon$.

LEMMA 4.8. *Let* H, S, *and* ε *be as in Lemma* 4.7 *and let* T *be a finite group. If* S *is finite,* $S = S^{-1}$, *and* $\phi\colon H \to T$ *is a group homomorphism,* $\phi(H) = T$ *then the Cayley graph* $G = G(T, \phi(S))$ *is a* $(|T|, |S|, \varepsilon)$ *enlarger.* ($\phi(S)$ *denotes here the multiset of cardinality* $|S|$; $\{\phi(s)\colon s \in S\}$.)

Proof. Clearly G is $|S|$-regular on $|T|$ vertices. We must show that $\lambda_1(G) \geqslant \varepsilon$. Let V be a vector space of dimension $|T|$, with a basis $(e_t)_{t \in T}$ indexed by the elements of T. Let π denote the regular (left) representation of T. Thus for $t \in T$, $\pi(t)$ is the permutation matrix given by

$$\begin{aligned}(\pi(t))_{w,u} &= 1 \qquad \text{if } w \cdot u^{-1} = t,\\ &= 0 \qquad \text{otherwise.}\end{aligned}$$

Therefore, the matrix $Q = Q_G$ (defined in Sect. 2), of the Cayley graph G is $|S| \cdot I - \sum_{s \in S} \pi \cdot \phi(s)$, where I is a $|T| \times |T|$ identity matrix. $\pi \cdot \phi$ is clearly a unitary representation of H. Let W denote the subspace of V consisting of all vectors the sum of whose coordinates is zero. Clearly W is invariant under $\pi \cdot \phi$ and thus $\pi \cdot \phi^w$ is a unitary representation of H in W. We claim that it is essentially nontrivial. Indeed, if $v \in V$ satisfies $\pi \cdot \phi(h)\, v = v$ for all $h \in H$ then, since $\phi(H) = T$, all coordinates of v are equal. Thus if, in addition, $v \in W$ then $v = 0$. Therefore $\pi \cdot \phi^w$ satisfies the hypotheses of Lemma 4.7 and hence for every unit vector $y \in W$ there exists an $s \in S$ such that $|(\pi \cdot \phi^w(s)\, y, y)| < 1 - \varepsilon$. This shows that

$$\min\{(Qy, y)\colon y \in W,\ \|y\| = 1\} > \varepsilon.$$

However, by Rayleigh's principle, the left-side of the last inequality is precisely $\lambda_1(G)$. This completes the proof. ∎

An immediate corollary of Lemma 4.8 is

THEOREM 4.9. *Let* H *be a discrete group having property* (T) *and let* S *be a finite set of generators of* H, $S = S^{-1}$. *Let* $\phi_i\colon H \to^{on} T_i$ *be group homomorphisms, where* $|T_i| \to \infty$, *and put* $G_i = G(T_i, \phi_i(S))$. *Then* G_i *is a family of linear enlargers, i.e., there exist fixed* $\varepsilon > 0$ *and* $k = |S|$, *such that* G_i *is a* $(|T_i|, k, \varepsilon)$ *enlarger.*

There are many known examples of groups H that enable us to apply Theorem 4.9. Here we point out one infinite family of such examples. As usual, let $SL(n, Z)$ denote the discrete group of all $n \times n$ matrices over the integers Z with determinant 1. Kazhdan [23] showed that for $n \geqslant 3$

$SL(n, Z)$ has property (T). In [30] it is shown that $SL(n, Z)$ has the following set of two generators:

$$B_n = \left\{ \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ (-1)^{n-1} & 0 & 0 & \cdots & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix} \right\}.$$

Define $S_n = B_n \cup B_n^{-1}$. Note that for $n \geqslant 3$ $|S_n| = 4$. Let $SL(n, Z_i)$ denote the group of all $n \times n$ matrices over the ring of integers modulo i with determinant 1, and let $\phi_i^{(n)}: SL(n, Z) \to SL(n, Z_i)$ be the homomorphism defined by $\phi_i^{(n)}((a_{rs})) = (a_{rs}(\text{mod } i))$. Clearly all the assumptions of Theorem 4.9 are satisfied by $H = SL(n, Z)(n \geqslant 3)$, $S = S_n$, $\phi_i = \phi_i^{(n)}$, and $T_i = SL(n, Z_i)$. This supplies an infinite number of explicit families of $(n, 4, \varepsilon)$-enlargers and thus, by Theorem 4.3, also an infinite number of families of linear expanders.

5. A Possible Application to Combinatorial Group Theory

Let H be a finite group and let γ be a set of generators of H, $\gamma = \gamma^{-1}$, $1 \in \gamma$. Define $k = k(H, \gamma)$ as the minimal integer such that $\gamma^k = H$, i.e., every element of H is a product of (at most) k members of γ. There are many results related to the determination or estimation of $k(H, \gamma)$.

Define $\delta = \gamma - \{1\}$ and let $G = G(H, \delta)$ be the corresponding Cayley graph. Put $\lambda_1 = \lambda_1(G)$. Note that $k(H, \gamma)$ is just the diameter of G and thus the following is an immediate consequence of Theorem 2.7.

Proposition 5.1.

$$k(H, \gamma) \leqslant 2[\sqrt{2|\delta|/\lambda_1} \log_2 |H|].$$

This result is, in some sense, best possible. Indeed, Combining the results of the previous section with the argument of Remark 2.8 one can prove

Proposition 5.2. *Let H be a discrete group having property (T) and let γ be a finite set of generators of H, $\gamma = \gamma^{-1}$, $1 \in \gamma$. Then there exist two positive constants $c_1 = c_1(H, \gamma)$ and $c_2 = c_2(H, \gamma)$ such that for any group homomorphism $\phi: H \to^{on} T$*

$$c_1 \log_2 |T| \leqslant k(T, \phi(\gamma)) \leqslant c_2 \cdot \log_2 |T|. \tag{5.1}$$

Thus the upper bound in (5.1), implied by Proposition 5.1, is, in a sense, best possible. In the previous section there are explicit examples of H and γ for which the last proposition can be applied.

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